

# Two twistor descriptions of the isomonodromy problem

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## Abstract

The connections between Hitchin and Mason's twistor descriptions of the isomonodromy problem are explored.

## 1 Introduction

Twistor theory was first explored by Penrose in his investigation of the role of holomorphicity and conformal symmetry in relativistic quantum field theory. The primary aim, to find a new route to the quantization of gravity, has yet to be fully achieved; but through the work of Ward, Hitchin, Mason, and others, the underlying geometry has provided a unifying framework for the study of integrable systems (see [4] for a review). It sheds light on the connections between

- integrable systems of partial differential equations;
- real and complex geometries with symmetry; and
- isomonodromic families of ordinary differential equations.

In this note, I shall concentrate on the third of these, and will explain the connections between two different twistor representations of isomonodromy, due, respectively, to Hitchin and Mason. The former construction links isomonodromy to problems in differential geometry; the latter gives a tool for the systematic study of the way in which the isomonodromic deformation equations arise from the dimensional reduction of integrable systems. The route from the first construction to the second has not appeared elsewhere.

## 2 Conformal reductions of the self-dual Yang-Mills equations

Ward showed [6] that there is a correspondence between, on the one hand, anti-self-dual solutions to the Yang-Mills equations on a suitable region of complex space-time and, on the other, holomorphic vector bundles over a corresponding subset  $U \subset \mathbb{CP}_3$  (complex projective 3-space). The Yang-Mills field is a connection  $D$  on a trivial bundle and the anti-self-duality condition is that its curvature should vanish on a special three-dimensional family of null 2-planes ( $\alpha$ -planes). The set of  $\alpha$ -planes is the *twistor space* of complex space-time—it is identified with a subset of  $\mathbb{CP}_3$ , in a way that is natural in the sense that the action of the proper conformal group in space-time corresponds to the action of the isomorphic group  $\mathrm{PGL}(4, \mathbb{C})$  on  $\mathbb{CP}_3$ . Ward’s construction maps  $D$  to the vector bundle  $E \rightarrow U$  whose fibres are the spaces of solutions to the linear equations  $Ds = 0$  over the  $\alpha$ -planes. The remarkable and non-trivial fact is that the construction is reversible:  $D$  can be uniquely recovered from  $E$ , with no other data required—a beautiful example of Penrose’s idea that relativistic field equations should reduce to holomorphicity conditions in twistor space.

The anti-self-duality condition is preserved by proper conformal transformation, and so it makes sense to look for solutions that are invariant under subgroups of the conformal group. The twistor construction then gives a correspondence between *conformal reductions* of the anti-self-duality condition and *equivariant vector bundles* on twistor space—that is, holomorphic bundles that are unchanged by the action of the corresponding subgroup of  $\mathrm{PGL}(4, \mathbb{C})$ . If the subgroup is  $m$ -dimensional and acts freely on an open subset of  $\mathbb{CP}_3$ , then the reduced system has  $4 - m$  independent variables. The one-dimensional examples give various monopole equations and the two-dimensional ones lead to a variety of familiar and widely-studied integrable systems—the KdV equation, the nonlinear Schrödinger equation, the Ernst equation, and many others. The three-dimensional ones give systems of ODEs with the Painlevé property. One thus sees in the twistor geometry a very direct connection between integrability of systems of partial differential equations and the Painlevé property of the systems of ODEs derived from them by dimensional reduction.

The Yang-Mills twistor construction encompasses only one special class of integrable systems. But the general ideas extend to include others—whether or not they can be taken far enough to include *all* integrable systems is an open question, and one that is unlikely to be answered so long as ‘integrability’ and ‘twistor’ retain their current elasticity of meaning. Whatever form the extension takes, however, one expects to see systems of

ODEs with the Painlevé property at the foot of any chain of dimensional reductions. One reason for this is the connection between the Painlevé property and isomonodromy, and the existence of a very general geometric construction for isomonodromic families of ODEs.

### 3 Equivariant bundles and isomonodromy

Suppose that we are given

- a complex manifold  $\mathcal{Z}$  and a complex Lie algebra  $\mathfrak{g}$  of the same dimension that acts on  $\mathcal{Z}$ , with the action being free on the complement of a hypersurface  $\Sigma \subset \mathcal{Z}$ ;
- a  $\mathfrak{g}$ -equivariant holomorphic principal bundle  $P \rightarrow \mathcal{Z}$  with structure group  $\hat{G}$ ;
- an embedded copy  $X \subset \mathcal{Z}$  of  $\mathbb{CP}_1$  which intersects  $\Sigma$  transversally, and which has the properties that  $P|_X$  is trivial and that

$$H^0(N, X) \neq 0, \quad H^1(N, X) = 0,$$

where  $N$  is the normal bundle of  $X$  in  $\mathcal{Z}$ . An action of  $\mathfrak{g}$  on  $\mathcal{Z}$  is a Lie algebra homomorphism into the holomorphic vector fields on  $\mathcal{Z}$ ; the action is free at a point  $z$  if corresponding map  $\mathfrak{g} \rightarrow T_z\mathcal{Z}$  is injective, and therefore, on dimensional grounds, an isomorphism. Each element of  $\mathfrak{g}$  determines a holomorphic vector field  $Y$  on  $\mathcal{Z}$ . When  $P$  is equivariant, then these vector fields in turn lift to vector fields on  $P$  that are preserved by the action of  $\hat{G}$ . In a local trivialization, the lift is given by a linear map  $Y \mapsto \theta_Y$ , where  $\theta_Y$  is a function on  $\mathcal{Z}$  with values in the Lie algebra  $\hat{\mathfrak{g}}$ , with the property

$$Y(\theta_{Y'}) + Y'(\theta_Y) + [\theta_Y, \theta_{Y'}] = 0$$

for every pair of generators  $Y, Y'$  of the  $\mathfrak{g}$  action. Under gauge transformations of the local trivialization,  $\theta_Y \mapsto h^{-1}\theta_Y h + h^{-1}Y(h)$ , where  $h$  takes values in  $\hat{G}$ .

By Kodaira's theorem [3],  $X$  is one of family of curves  $X_m \subset \mathcal{Z}$  that intersect  $\Sigma$  transversally, labelled by a parameter space  $M$  of dimension  $H^0(X, N)$ ; for almost every  $m \in M$ , the restricted bundle  $P|_{X_m}$  is trivial—although this does not imply that  $P$  itself is trivial. The *jumping lines* are the isolated members of the family for which  $P|_{X_m}$  is nontrivial.

Given these, we construct an isomonodromic family of ODEs as follows. First, we note that the action of  $\mathfrak{g}$  on  $P$  determines a flat  $P$ -connection  $D$  on the complement of  $\Sigma$ ,

characterized by  $D_Y = d + \theta_Y$  for each generator  $Y$ . Next,  $P$  and the connection are pulled back to the *correspondence space*  $\mathcal{C}$ . This is the space whose points are pairs  $(z, X_m)$ , with  $X_m$  one of the family of curves and  $z \in X_m$ . If we exclude the jumping lines from  $M$ , then the pull-back of  $P$  is the trivial bundle, and the pull-back of the connection is flat and meromorphic: it is singular where  $z \in \Sigma$ .

The correspondence space is fibred over  $\mathcal{Z}$  by  $(z, X_m) \mapsto z$ , and over  $M$  by  $(z, X_m) \mapsto m$ ; the fibres of the second fibration are the curves  $(z, X_m)$ , with  $m$  fixed, which are all copies of  $\mathbb{CP}_1$ . In the global trivialization of the pulled-back bundle over  $\mathcal{C}$ , the restriction of the connection to one of these fibres is of the form

$$d - A(\zeta) d\zeta$$

where  $\zeta$  is a stereographic coordinate and  $A$  is a rational function on  $\mathbb{CP}_1$  with values in  $\hat{\mathfrak{g}}$ —the Lie algebra of  $\hat{G}$ . It has poles at the points where  $X_m$  meets  $\Sigma$ . Thus we have a family of linear ODEs, labelled by points of  $M$

$$\frac{dy}{d\zeta} = Ay$$

Here  $y$  takes values in a representation space of  $\hat{G}$ . The ODEs are uniquely determined by the data up to conjugation of  $A$  by a holomorphic map  $h : M \rightarrow \hat{G}$ .

If the coordinate is chosen so that  $\zeta = \infty$  is not an intersection point with  $\Sigma$ , then  $A$  has a zero of order 2 at infinity. A pole of order  $r+1$  in  $A$  is a singularity of rank  $r$  in the linear system.

The solutions to the ODE are parallel sections of the associated vector bundle over the lines  $X_m$ —although these exist only locally and are not single-valued in the large. Isomonodromy follows more or less directly from the fact that  $D$  is the restriction of a flat meromorphic connection. If all the poles of  $A$  are simple, then ‘isomonodromic’ means no more than that the monodromy representation is constant up to conjugacy as  $m$  varies; if there are poles of higher order, then it involves in addition the preservation of other data associated with the ODEs. In either case, the monodromy representation coincides with the holonomy of the flat connection on  $P$ .

The archetypical example is the sixth Painlevé equation [5]. This we obtain by taking  $\mathcal{Z}$  to be a neighbourhood of a line in  $\mathbb{CP}_3$  and  $\mathfrak{g}$  to be the diagonal subalgebra of the Lie algebra of the projective general linear group  $\mathrm{PGL}(4, \mathbb{C})$ . The solutions to  $P_{VI}$  correspond to the equivariant  $\mathrm{SL}(2, \mathbb{C})$ -bundles over  $\mathcal{Z}$ .

## 4 Local form of the connection

In a local trivialization of  $P$ ,

$$D = d - \alpha \quad (1)$$

where  $\alpha$  is a meromorphic 1-form on  $\mathcal{Z}$  with values in the  $\hat{g}$ . It is nonsingular on the complement of  $\Sigma$  and satisfies the flatness condition

$$d\alpha - [\alpha, \alpha] = 0.$$

Its restriction to a curve  $X_m$  is gauge-equivalent to  $A d\zeta$ .

We shall make two ‘genericity’ assumptions. The first is that there is an abelian subalgebra  $\mathfrak{t} \subset \hat{g}$  such the leading coefficient in  $A$  is conjugate to an element of  $\hat{g}$  at each pole. When  $\hat{G}$  is the general linear group and  $\mathfrak{t}$  is the diagonal subalgebra, this is a consequence of the standard genericity assumption that the eigenvalues of the leading coefficients are distinct. The second assumption is that there is a curve in the family through every point of  $\mathcal{Z}$ .

Let  $a \in \Sigma \cap X$ . By the second assumption, we can identify a neighbourhood of  $a$  in  $\mathcal{Z}$  with a neighbourhood of  $(a, 0)$  in  $\Sigma \times \mathbb{C}$  the curves  $\zeta \mapsto (z, \zeta)$ , with  $z$  fixed, are parts of curves in the family. Suppose that  $A$  has a pole of order  $r + 1$  at  $a$ . Then, by the first assumption and the flatness condition,

$$h\alpha h^{-1} = \frac{a d\zeta + \zeta \beta}{\zeta^{r+1}} + \gamma,$$

where  $h$  takes values in  $\hat{G}$  and is holomorphic,  $a$  takes values in  $\mathfrak{t}$ ,  $\beta$  and  $\gamma$  are holomorphic at  $\zeta = 0$ ,  $\beta$  has no  $d\zeta$  component, and

$$d_z a = -r\beta + O(\zeta^{r+1}),$$

where  $d_z$  denotes the exterior derivative with  $\zeta$  held fixed. By expanding  $a$  in powers of  $\zeta$ , we deduce that

$$h\alpha h^{-1} = d\tau + m d \log \zeta + \gamma', \quad (2)$$

where  $\tau = p/\zeta^r$ ,  $p$  a  $\mathfrak{t}$ -valued polynomial in  $\zeta$ ,  $m$  is a constant element of  $\mathfrak{t}$ , and  $\gamma'$  is holomorphic at  $\zeta = 0$ . The constant  $m$  is the ‘exponent of formal monodromy’, and the restriction of  $\tau + m \log \zeta$  to a one of the curves of the family is the diagonal exponent in the formal solution of the linear system in [2].

If  $Y$  is one of the generators of the action of  $\mathfrak{g}$ , then  $i_Y \alpha$  is constant and  $Y(\zeta) = 0$  on  $\Sigma$ . So, despite the fact that  $\tau$  blows up as  $\zeta^{-r}$ , its derivative  $Y(\tau)$  is holomorphic at  $\zeta = 0$ .

## 5 Two constructions

In general, the information contained in the linear system of ODEs and its deformations is contained in the action of  $\mathfrak{g}$  on both the base space  $\mathcal{Z}$  and the bundle  $P$ . At the extremes are two special constructions.

### First construction

In the first, due to Hitchin [1],  $P$  is trivial and all the data are encoded in the geometry of  $\mathcal{Z}$  and  $\mathfrak{g}$ . Hitchin's construction has been exploited to generate interesting geometries by using Penrose's nonlinear graviton construction, and its variants. With  $\mathfrak{g} = \text{sl}(2, \mathbb{C})$ , for example,  $\mathcal{Z}$  is the twistor space of a four-dimensional complex Riemannian manifold with anti-self-dual conformal structure and  $SL(2, \mathbb{C})$  symmetry (a ‘Bianchi IX’ geometry). It is shown in [7] that if  $\mathfrak{g}$  is the Lie algebra of the general linear group and if  $A$  satisfies the standard condition that its leading coefficients at its irregular singularities have distinct eigenvalues, then the corresponding isomonodromic family can be obtained from such a twistor space.

### Second construction

At the other extreme,  $\mathcal{Z}$  is a standard space that carries only information about the number of singularities in the linear system and their ranks. The particular isomonodromic family is encoded in the bundle  $P \rightarrow \mathcal{Z}$ . Such a standard twistor space can be constructed as follows. Let  $\mathfrak{t}$  be an abelian subalgebra of  $\hat{\mathfrak{g}}$ —we shall keep in mind the main example in which  $\mathfrak{t}$  is the Lie algebra of the diagonal subgroup of the complex general linear group. Let  $r$  be a nonnegative integer. If  $r = 0$ , let  $H_r$  denote the abelian group  $\mathbb{C}^*$  (the multiplicative group of complex numbers); if  $r > 0$ , let  $H_r = \mathbb{C}^* \times \mathfrak{t}$ , with the group law

$$(\lambda, t) \cdot (\lambda', t') = (\lambda\lambda', \lambda^r t' + \lambda'^r t).$$

For  $r > 0$ , this acts linearly on the vector space  $V_r = \mathbb{C} \oplus \mathfrak{t}$  by

$$(Z, W) \in \mathbb{C} \oplus \mathfrak{t} \mapsto (\lambda Z, \lambda^r W + t Z^r).$$

For  $r = 0$ , we take  $V_0 = \mathbb{C}$ .

Denote by  $H$  the quotient group  $(H_{r_0} \times \cdots \times H_{r_n})/\mathbb{C}^*$ . The *standard twistor space*  $\mathcal{Z}_S$  associated with  $H$  is constructed from the linear action of the  $H_r$ s on

$$\mathbb{V} = V_{r_0} \oplus \cdots \oplus V_{r_n},$$

where the  $r_i$ s are the ranks of the singularities of the linear system. By picking out the  $Z$ s in each summand, we have a projection  $\pi : \mathbb{V} \rightarrow \mathbb{C}^{n+1}$ . We define  $\mathcal{Z}_S$  to be the quotient of

$$\mathbb{V} \setminus \pi^{-1}(0)$$

by the action of  $\mathbb{C}^*$ , and take  $G = H$ .

As a complex manifold,  $\mathcal{Z}_S$  is the total space of copies of the line bundles  $\mathcal{O}(r_i)$  over  $\mathbb{CP}_n$ , with the projection onto  $\mathbb{CP}_n$  given by  $\pi$ ; in the Fuchsian case, in which all the ranks are zero,  $\mathcal{Z}_S = \mathbb{CP}_n$ . The action of the  $H_{rs}$  on the  $V_{rs}$  gives an action of  $H$  on  $\mathcal{Z}_S$  which is transitive and free on the complement of the

$$\Sigma = \bigcup \pi^{-1}(\Pi_i)/C^*,$$

where the  $\Pi_i$ s are the coordinate hyperplanes in  $\mathbb{C}^{n+1}$ , and the curves  $X_m$  are the sections of  $\mathcal{Z}_S$  over lines in  $\mathbb{CP}_n$ .

Consider the form of  $\alpha$  in a neighbourhood of a point on  $\Pi_i$ . If  $r_i > 0$ , then the generators of the  $H$ -action are nonzero in the neighbourhood, except for those in the Lie algebra of the  $t$  factor of  $H_{r_i}$ . Denote by  $Z_i, W_i$  the corresponding homogeneous coordinates on  $\mathcal{Z}_S$ . By choosing the local trivialization of  $P$  to be invariant along the non-vanishing generators, we can ensure that  $\tau$  depends only on  $Z_i, W_i$ . Moreover, since it is constant along the non-vanishing generator of  $H_{r_i}$ , it depends on these variables only through the combinations  $W_i/Z_i^r$ . Consequently  $\tau = p/Z_i^r$ , where  $p$  is a linear function of  $W_i$ . By constant linear transformation of the variables  $W_i$ , therefore, we can ensure that  $\tau = W_i/Z_i^r$ . This relates the coordinates homogeneous  $W, Z$  on  $\mathcal{Z}_S$  to the deformation parameters in [2].

It is shown in [7] that if  $\hat{G}$  is the general linear group and if  $A$  satisfies the standard condition that its leading coefficients at its irregular singularities have distinct eigenvalues, then the corresponding isomonodromic family can be obtained from an equivariant bundle  $P \rightarrow \mathcal{Z}_S$ , at least locally, that is, by restricting  $\mathcal{Z}_S$  to an open neighbourhood of one of the curves  $X_m$ . The ranks of the singularities are the integers  $r_i$ .

The second twistor construction allows one to see, in a systematic way, how the different isomonodromic deformation equations fit into the hierarchies associated with standard integrable systems. In the archetypical example,  $n = 3$  and all the ranks are zero, so  $\mathcal{Z}_S = \mathbb{CP}_3$ .

## 6 From the second construction to the first

Suppose that we are given  $P \rightarrow \mathcal{Z}$  as in the first construction. The passage from the second construction to the first is the ‘switch map’ in [4], which interchanges the roles of

$\mathfrak{g}$  and  $\hat{\mathfrak{g}}$ . The total space of the principal bundle  $P \rightarrow \mathcal{Z}$  carries actions of the  $\hat{\mathfrak{g}}$  (the Lie algebra of its structure group) and of  $\mathfrak{g}$  (since it is equivariant). The two actions commute and their orbits are transverse except over  $\Sigma$ . The curves in  $\mathcal{Z}$  lift to  $P$  by the triviality condition. If we take a neighbourhood of such a lifted curve, and quotient by the action  $\hat{\mathfrak{g}}$ , then we obtain the twistor space of the second construction; the projections from  $P$  of the lifts of the curves in  $\mathcal{Z}$  are the twistor curves in the new twistor space.

## 7 From the first construction to the second

To go in the other direction, we start with  $\mathcal{Z}$  and the  $\mathfrak{g}$ -action of the first construction, with  $P$  is trivial. Then  $D$  is globally of the form (1).

The hypersurface  $\Sigma$  is made up of a number of connected components, each intersecting a twistor curve  $X$  at one of the singularities of the linear system. Let us concentrate for the moment on one component  $S$  on which  $\zeta = 0$ . We shall construct an equivariant bundle  $B_r \rightarrow \mathcal{Z}$  from  $S$  and an action of  $\mathfrak{g}$  on  $B_r$  that together encode the position of the corresponding pole of  $A$  along with information about the singular part of  $A$ . First we define  $L_r$  to be the line bundle with  $-S$ . This is equivariant because  $S$  is invariant. It has fibre coordinate  $z$  away from  $S$ , and fibre coordinate  $z'$  in a neighbourhood of  $S$ , with transition rule

$$z' = z/\zeta,$$

and a  $\mathfrak{g}$ -invariant section over the complement of  $S$  given by  $z = 1$ . This section determines the action  $\mathfrak{g}$  on the whole of  $B_r$  since  $Y(\zeta)/\zeta$  is nonsingular on  $S$ .

When  $r > 0$ , we also introduce an equivariant affine bundle  $T_r \rightarrow \mathcal{Z}$  with fibre  $\mathfrak{t}$  and transition rule

$$w = w' - \tau$$

$w, w' \in \mathfrak{t}$  are fibre coordinates. The action of  $\mathfrak{g}$  is determined away from  $S$  by the condition that  $z$  and  $w$  should be invariant. It extends holomorphically over  $S$  since if  $Y$  is one of the generating vector fields of the action of  $\mathfrak{g}$  on  $\mathcal{Z}$ , then

$$\frac{Y(\zeta)}{\zeta} \quad \text{and} \quad Y(\tau)$$

are nonsingular at  $\zeta \rightarrow 0$ .

In both cases, the action of  $\mathfrak{g}$  on  $L_r$  and on  $T_r$  commutes with the natural action of  $H_r$  given by

$$z \mapsto \lambda z, \quad w \mapsto w + t.$$

We construct such a line bundle  $L_{r_i}$  for each component of  $\Sigma$ , and affine bundle  $T \rightarrow \mathcal{Z}$  by taking the product of the  $T_r$ s for each component of  $\Sigma$  for which  $r > 0$ . We use a subscript  $i$  to denote the quantities associated with the  $i$ th component of  $\Sigma$ . Put

$$P = T \times (L_{r_0} \oplus \cdots \oplus L_{r_n})/\mathbb{C}^*,$$

where  $\mathbb{C}^*$  acts by  $z_i \mapsto \lambda z_i$ . The fibres of  $P$  are products of  $\mathbb{CP}_n$ , with homogeneous coordinates  $z_i$ , with  $n+1$  copies of the affine space modelled on  $\mathbf{t}$ . We can lift a twistor curve  $X \subset \mathcal{Z}$  to  $P$  by choosing a stereographic coordinate  $\lambda$  on  $X$ , and putting  $z_i : z_j = (\lambda - \lambda_i) : (\lambda - \lambda_j)$ , where the points  $\lambda = \lambda_i$  are the intersection points of  $X$  with  $\Sigma$ ; and by picking out the unique section of  $T|_X$  for which  $w_i - \tau_i$  is holomorphic at the intersection with  $S_i$ .

We then commuting actions of  $\mathfrak{g}$  and  $H$  on  $P$ . We also have a projection  $P \rightarrow \mathcal{Z}_S$  given by

$$(z_i, w_i) \mapsto (Z_i, W_i) = (z_i, z_i^r w_i)$$

which extends holomorphically to the fibres of  $P$  over  $\Sigma$  since  $z = \zeta z'$  and  $z^r w = z'^r \zeta^r (w' - \tau)$  are holomorphic at  $\zeta = 0$ . Moreover, if we lift a twistor curve to  $P$  and then project it into  $\mathcal{Z}_S$ , then we obtain one of the twistor curves in  $\mathcal{Z}_S$ .

The quotient  $P = B/\mathbb{C}^*$  by the subgroup  $\mathbb{C}^* \subset H$  is part of the total space of a principal bundle with structure group  $H/\mathbb{C}^*$  over a neighbourhood  $U$  of a twistor curves in the standard twistor space. From it, we recover the whole of  $P$  over  $U$ , and thus the second description of the isomonodromic family.

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